Computer-drawn field lines and potential surfaces for a wide range of field configurations

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An interactive computer program has been written that computes field lines and equipotential surfaces for a wide range of field configurations. The mathematical techniqûe and details of the program, the input data, and different modes of graphical representation are described, and various examples are presented. Possible extensions of the method are discussed.

I. THE PROBLEM

The use of field lines and equipotential surfaces for the visualization of vector fields (electrostatic and magnetostatic fields, stationary liquid and heat flow fields, stationary diffusion fields, etc.) is well known. However, the graphical representations, usually found in textbooks, are restricted to very simple examples, and often they are only qualitative. Quantitative pictures of more complex examples, which are highly desirable for didactical purposes, are rare. Furthermore, it would be very helpful for students if they could visualize themselves fields of their own choice.

We present here a technique that has been developed into an interactive computer program. It generates graphs of fields and potentials in two or three dimensions for arbitrary configurations of point sources, vortices, dipoles, and homogeneous fields.

In the following we first discuss the mathematical technique used to compute the field lines. We then quickly review the properties of stationary fields, especially those of simple singularities (sources and vortices), using simple continuity arguments. A short description of the program itself and especially of the necessary input data precedes the presentation of a number of examples which form the essential part of this paper.

II. MATHEMATICAL APPROACH

It is no problem to obtain the field vector or the potential value at any point, since the field or potential of each individual charge or vortex is known and they have simply to be added according to the superposition principle. The actual problem is to fill all space with suitably spaced field lines and equipotential surfaces.

Characteristic for a field line is that its tangent is parallel to the field vector, say $\mathbf{E}$, at all points $\mathbf{x}$. Let a field line be represented by the function

$$\mathbf{x} = \mathbf{x}(t), \tag{1}$$

where t is a scalar parameter. Therefore

$$\frac{d\mathbf{x}}{dt} = f\mathbf{E}. \tag{2}$$

The proportionality factor f has to be the same for all three components of $\mathbf{x}$, but it need not be constant, since only the tangent vector $d\mathbf{x}/dt$ is important. Usually the magnitude of $d\mathbf{x}/dt$ is chosen to be unity, so that Eq. (2) reads[1]

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{E}}{|\mathbf{E}|} = \frac{\mathbf{E}}{(\mathbf{E}^2)^{1/2}}. \tag{3}$$

We prefer, however, another choice of the proportionality factor f, namely one that allows to identify the parameter t with the potential ϕ. We expect that there exists a function $f(E)$ such that

$$\frac{d\mathbf{x}}{d\phi} = \mathbf{E}f(E) \tag{4}$$

holds. Since we move along the field line where $\mathbf{E}$ and $d\mathbf{x}$ are parallel, we have

$$d\phi = \mathbf{E}\cdot d\mathbf{x} = E\,dx,$$

where dx stands for $|d\mathbf{x}|$.

Introducing this into (4) yields

$$\frac{dx}{E\,dx} = E\,f(E), \qquad \text{i.e.,} \qquad f(E) = \frac{1}{E^2},$$

and finally

$$\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{d\phi} = \frac{\mathbf{E}}{E^2}. \tag{5}$$

If we have a computer program that integrates ordinary differential equations in time and that sets marks after equal time intervals, it can be used to compute a field line and mark equal potential intervals. A field line with potential marks determines the complete field vector since the magnitude of the field strength is inversely proportional to the distance Δs between two potential marks:

$$E = \Delta\phi/\Delta s = \text{const}/\Delta s. \tag{6}$$

Further advantages of Eq. (5) with respect to Eq. (3) are that there is no need to compute a square root and that, if the integration is performed in equal steps of t, the corresponding steps in $\mathbf{x}$ automatically are small at regions of high field and large where the field is small. This is desirable from the point of view of numerical accuracy and the efficient drawing of smooth curves.

III. SHORT REVIEW OF THE PROPERTIES OF SOURCES AND VORTICES

Although electrostatic fields **E** are usually more extensively studied than the stationary velocity field **v** of an incompressible frictionless liquid, the latter is much better suited for many didactic arguments because it is more closely connected to experience.

We use it to review briefly the properties of a single source and a single vortex.

A. General properties derived from continuity relations

The liquid flows away from the source along the field lines of the velocity field. If we define a *flow tube* as bounded by field lines, the liquid flows along such tubes. Liquid can enter or leave the tubes only through sources and sinks (sources with negative strength). Since the field lines are orthogonal to equipotential surfaces, these surfaces intersect the flow tubes at right angles. A flow tube and three sections A_i, A_{i+1}, A_{i+2} of equipotential surfaces corresponding to the potentials ϕ_i, ϕ_{i+1}, ϕ_{i+2} are shown in Fig. 1(a). Since the liquid is incompressible, the volume flowing through the three surfaces per unit time must be equal:

$$A_i v_i = A_{i+1} v_{i+1}, \qquad v_i/v_{i+1} = A_{i+1}/A_i. \tag{7a}$$

The field strength v_i is inversely proportional to the size of the corresponding cross section A_i of the equipotential surface. Since the number of field lines penetrating each surface section of the tube is constant, *the number of field lines per unit area of equipotential surface,* i.e., the field line density in space, is proportional to the field strength, provided that field lines are started with uniform density from an equipotential surface very close to the source.

Since—by construction—the potential difference between equipotential surfaces is constant and

$$v_i = -\text{grad}\phi \approx -\Delta\phi/l_i, \tag{8a}$$

where l_i is the spatial distance between the equipotential surfaces, one has

$$l_i/A_i = l_{i+1}/A_{i+1}. \tag{9a}$$

This means that the ratio between length and front surface of all pieces of a flow tube bounded by equipotential surfaces is constant.

It is very instructive also to consider the special case of two-dimensional (2D) fields, which can be realized by the frictionless flow of a liquid between two parallel glass plates. This case also represents those three-dimensional (3D) fields which are uniform in one direction, like the electric field of a cylindrical condenser. In the 2D case the flow tubes become plane surfaces bounded by field lines and pieces of equipotential lines as sketched in Fig. 1(b). With the nomenclature of that figure one easily finds the relations

$$v_i/v_{i+1} = d_{i+1}/d_i, \tag{7b}$$

$$v_i \approx \Delta\phi/d_i, \tag{8b}$$

and

$$l_i/d_i = l_{i+1}/d_{i+1}. \tag{9b}$$

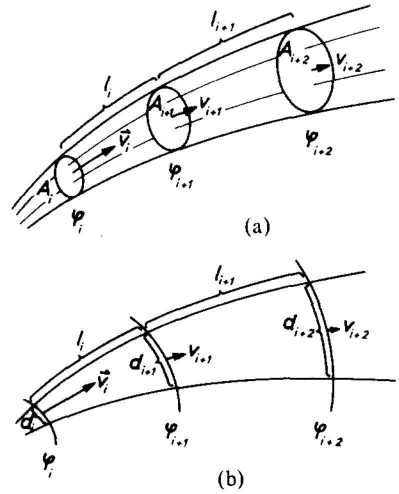


Fig. 1. Flow tube in the stationary velocity field of an incompressible liquid: (a) three-dimensional (3D) field; (b) two-dimensional (2D) field.

The field strength is now proportional to the *number of field lines per unit length* of equipotential line, i.e., to the field line density in the plane, provided the field lines were started equidistantly from an equipotential line very close to the pole. All quadrangles formed by field lines and equipotential lines are similar. For the special case $l_i = d_i$ they are squares. A square mesh is obtained for the case that n field lines emanate from a source of strength q by choosing $\Delta\phi = q/n$.

B. Field of a single source

A source is a point emitting a constant volume q (called the strength of the source) of liquid per unit time. (For negative q one speaks of a sink.) For symmetry reasons the field lines are straight lines pointing away from the source, and the equipotential surfaces are spheres around the source. Since every sphere is penetrated by the volume q per unit time, we have $q = v(r)\cdot 4\pi r^2$, i.e.,

$$v = q/4\pi r^2, \qquad \mathbf{v} = (q/4\pi r^3)\mathbf{r}. \tag{10a}$$

The potential is

$$\phi = \int_\infty^r \mathbf{v}(r')\cdot d\mathbf{r}' = \int_\infty^r v(r')\,dr' = \frac{q}{4\pi r} \tag{11a}$$

if we set $\phi = 0$ for $r \to \infty$.

By analogous reasoning we find that in a 2D field the (2D) volume q per unit time penetrates every circle, i.e., $q = v(r)2\pi r$ or

$$v = q/2\pi r, \qquad \mathbf{v} = (q/2\pi r^2)\mathbf{r}, \tag{10b}$$

$$\phi = \int_1^r \mathbf{v}(\mathbf{r}')\,d\mathbf{r}' = \int_1^r v(r')\,dr' = (q/2\pi)\ln r, \tag{11b}$$

with the convention $\phi = 0$ at $r = 1$.

C. Field of a single straight vortex line

In a 3D field $\mathbf{v}(\mathbf{r})$, a line in space is called a vortex line if

$$\omega = \oint \mathbf{v}\cdot d\mathbf{s} \neq 0, \tag{12}$$

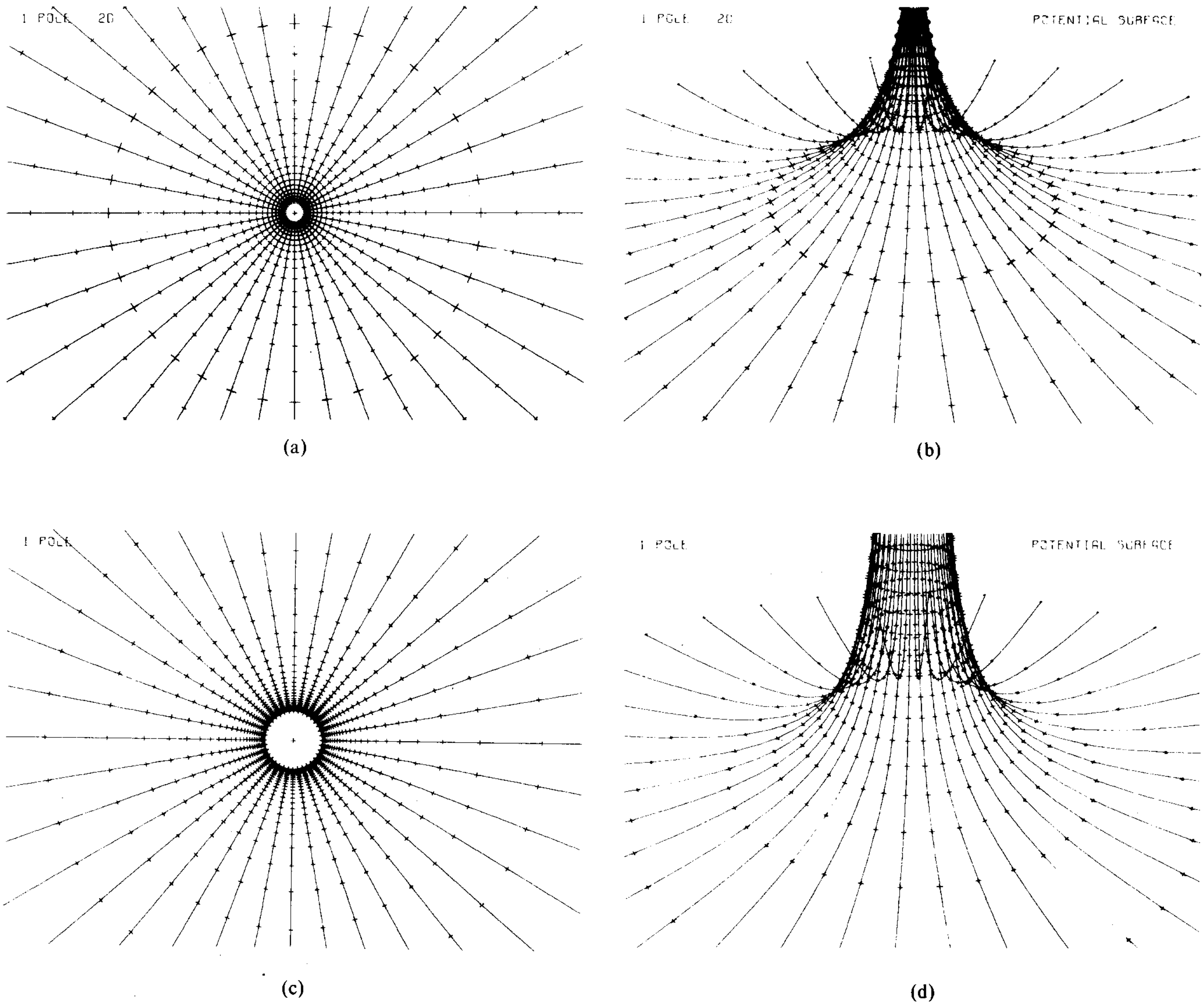


Fig. 2. Field of a single source of strength $q = 1$. (a) Field lines in the two-dimensional (2D) case. Equipotential marks are placed at intervals $\Delta\phi = 1/36$. The potential $\phi = 0$ is indicated by larger potential marks. (b) Corresponding potential surface. (c) Field lines in the three-dimensional (3D) case. Equipotential marks have intervals $\Delta\phi = 1/16\pi$. (d) Potential surface corresponding to (c).

where the integration is performed along any closed path which surrounds the line. If the space contains only one vortex line, the above integral is a constant called the vortex strength. This is in contrast to the field of a source, in which every such integral vanishes. We specialize to the case where the vortex line is straight. The field then must have a cylindrical symmetry with respect to it. The field lines are circles, because the liquid cannot flow away to infinity since no source is present that would yield new liquid. The integral (12) performed along a circle of radius r is $\omega = 2\pi r v$. Therefore

$$v = \omega/2\pi r, \qquad \mathbf{v} = (\boldsymbol{\omega} \times \mathbf{r})/2\pi r^2. \tag{13}$$

The vector $\boldsymbol{\omega}$ has the direction of the vortex line and the magnitude of the vortex strength.

In order to preserve the fact that the number of field lines per unit length of equipotential line is proportional to the field strength, the distance d_i between two field lines has to be proportional to r:

$$r_{i+1} - r_i = d_i = a r_i, \qquad r_{i+1}/r_i = \text{const.} \tag{14a}$$

Therefore the distance of field lines from the vortex should be chosen to increase in geometric progression; i.e., the logarithm should increase linearly, since

$$\ln r_{i+1} - \ln r_i = \text{const}'. \tag{14b}$$

If the vortex line coincides with the z axis, the potential difference between two points with radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ and azimuth angles α_0 and α_1 in cylindrical coordinates is

$$\phi = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{v} \cdot d\mathbf{r} = r \int_{\alpha_0}^{\alpha_1} v \, d\alpha = \left(\frac{\omega}{2\pi}\right)(\alpha_1 - \alpha_0).$$

With the convention $\phi = 0$ for $\alpha_0 = 0$ the potential of a point with azimuth α is

$$\phi = (\omega/2\pi)\alpha. \tag{15}$$

The potential is simply proportional to the point's azimuth angle. However, as a many-valued function of the azimuth, it depends on the number of turns performed around the vortex line [cf. Fig. 3(b)].

IV. PROGRAM AND INPUT DATA

The integration of Eq. (5) and the graphical presentation of the results are performed using the interactive program "particles and fields," developed originally to compute and display particle trajectories in fields.[2] The input data to the program have the form of a matrix of numbers. They can be classified into the three groups: *field data, field line data,* and *representation data.*

A. Field data

The field is determined by several rows of input data. One row is required for (a) every point charge (charge and position), (b) every vortex line (vortex strength, position, and orientation in space), (c) every dipole (vector of dipole moment and position), and (d) an external homogeneous field (constant field vector). The zero level of the potential is fixed in the program by the following convention. The potential of a single source is set to zero at unit distance in the case of 2D fields and at infinity for 3D fields. The potential of a vortex is equal to the product of vortex strength and azimuth angle. The total potential of several sources, vortexes, and an external field is found by addition.

B. Field line data

Once the field is given, a field line is uniquely determined by one point on the field line. For the purpose of graphical representation an initial potential ϕ_i and a final potential ϕ_f are specified, which are used for a whole bundle of field lines. Every line in the bundle is characterized by its starting point. If the potential ϕ_s of the starting point is outside the region between ϕ_i and ϕ_f, the field line is drawn from ϕ_i to ϕ_f; otherwise, it is drawn from ϕ_s to ϕ_f.

The program generates groups of starting points corresponding to bundles of field lines. According to the symmetries discussed in Sec. III the following types of groups are available: (a) for every point charge the field lines are started evenly spaced on a small circle around the charge; (b) for every dipole the field lines are started evenly spaced on two very small circles centered on the dipole axis and touching each other at the dipole position; (c) for every vortex the field lines are started on the straight line originating from the vortex, the distances from the vortex increasing in geometric progression; (d) for a homogeneous field the starting points are placed equidistantly on a straight line which should be perpendicular to the field direction and far from any singularities.

C. Representation data

By the integration one obtains for every field line a sequence of quadruplets of coordinates

$$x_k, y_k, z_k, \phi_k, \qquad k = 1, 2, \ldots, N.$$

The field lines can be considered as lines in a four-dimensional (4D) space spanned by x, y, z, and ϕ. The program provides three different possibilities of projecting these lines onto the 2D paper plane. Let us consider the projection to proceed in two steps: first into a 3D subspace S, and then onto the paper plane. If S is the x,y,z space, the resulting picture will contain the usual field lines.

Two cases can be distinguished:

(a) All field lines stay in the same plane. Then this plane is chosen to be the paper plane [examples in Figs. 2(a) and 2(c)].
(b) The field lines do not stay within one plane, but are projected perspectively onto a plane [examples in Figs. 7(a) and 7(b)].

If all the field lines are confined to one plane, one can span the space S by this plane and the potential. All the field lines in S are then contained in a 3D surface, the *potential surface.* Equipotential lines are contour lines, and the field lines are lines of steepest descent on the potential surface. This surface can again be projected perspectively onto the paper plane. Examples of this mode of projection are Figs. 2(b) and 2(d).

D. Off-line version of the program

An off-line version of the computer program is available. It runs on an IBM 360/65 computer. The input data are provided on punched cards. The output is drawn on a Calcomp plotter.

The program consists of the following parts: (a) decoding of input data; (b) control of the starting positions of the field lines; (c) integration of a field line in space—the field lines are stored as x,y,z,ϕ-coordinate quadruplets of closely neighboring points; (d) projection of these points in the desired mode onto the paper plane, and drawing of this projection.

E. Interactive version of the program

Easier to handle and well suited for student's use is the interactive version of the program which is currently implemented on a PDP-11/45 computer. The user inputs or modifies the data on a keyboard of a graphic terminal (Tektronix 4012 storage tube). The output is presented on the terminal's screen. In this way one can quickly change parameters and study the result. Permanent copies can be made using the Calcomp plotter or a special hardcopy unit accompanying the storage tube.

F. Portability of the program

The program[2] is written entirely in FORTRAN IV and structured so that it can be run using a storage overlay. There should be no major difficulty implementing it on other computers with equivalent graphical output facilities, at least 64 k words of storage and a disk.

V. EXAMPLES

We illustrate the use of the program by a number of examples, beginning with very simple problems and proceeding to more complex ones.

A. Single sources and vortices

The field lines of a 2D field in the x,y plane generated by a single source are shown in Fig. 2(a). For fixed equidistant

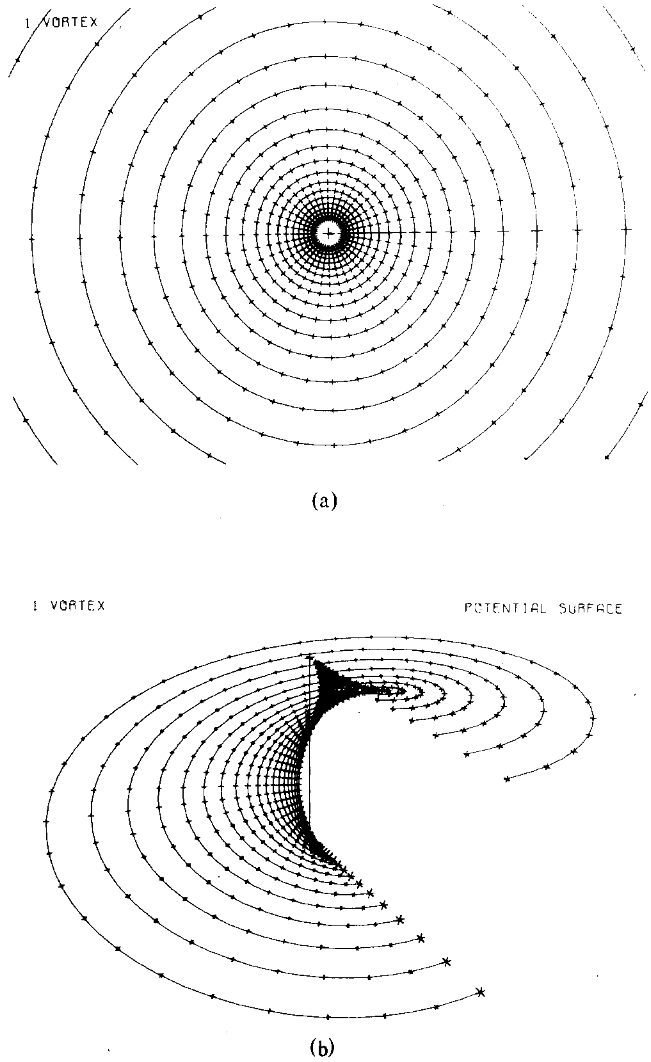


Fig. 3. Field of a single vortex of strength $\omega = 1$ situated on the z axis. (a) Field lines in the x,y plane. Interval between potential marks is $\Delta\phi = 1/36$. (b) Potential surface corresponding to (a). The observer's azimuth angle is $\alpha_{obs} = -34°$. The potential is compressed with respect to Figs. 2(b) and 2(d); a relative scaling factor $F_\phi = 1/8$ was used.

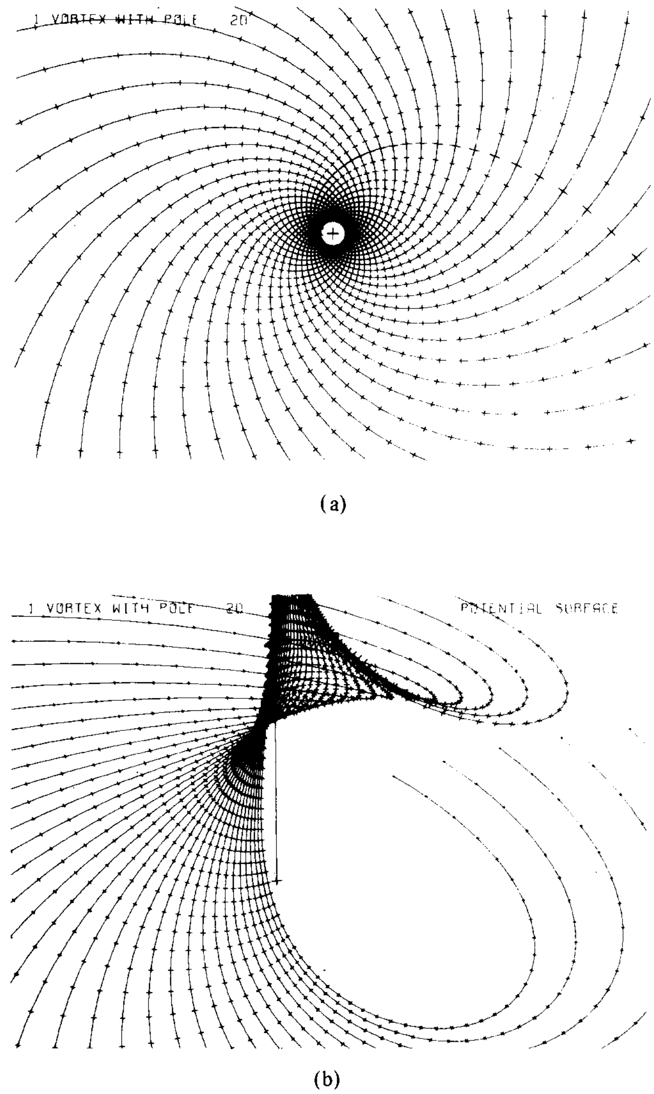


Fig. 4. Two-dimensional (2D) field of a source ($q = 1$) and a vortex ($\omega = 1$) both in the origin of the x,y plane. (a) Field lines $\Delta\phi = 1/36$. (b) Potential surface, $F_\phi = 1/4$. In x,y projection the outermost field line at the upper right would coincide with the innermost field line at the bottom.

potentials ($\Delta\phi$ = const) crosses are placed on the field lines. They serve a double purpose. Their density indicates the field strength. Corresponding potential marks on neighboring field lines, if joined by a continuous curve, form equipotential lines. The equipotential line $\phi = 0$ is marked by larger crosses in all figures. One easily recognizes that all quadrangles formed by neighboring field lines and equipotential lines are similar (in fact, they are squares), as described in Sec. III A.

The potential surface corresponding to these field lines is shown in Fig. 2(b), which is a perspective view of the field lines in x,y,ϕ space. (For all pictures of potential surfaces in this paper the observer's optical axis is inclined downwards. It forms an angle of 30° with the x,y plane. Apart from the double-sized marks of the zero potential, all potential marks have the same size in space. Hence, their apparent size in the perspective projection is a useful clue to depth.)

Next we consider the 3D field of a single source. The source is situated in the origin. The field lines in the x,y plane are shown in Fig. 2(c). Since neither the field lines nor the equipotential marks can be resolved optically near the pole, a circle has been left empty in that region. The quadrangles formed by field lines and equipotential lines are no longer similar. The potential surface is shown in Fig. 2(d). In comparison with Fig. 2(b) it is much steeper at small distances and much flatter at large distances from the pole [cf. Eqs. (10)].

The field of a single vortex is illustrated in Fig. 3. Field lines are shown in Fig. 3(a), and potential surface for the azimuth range $0 < \alpha < 2\pi$ is shown in Fig. 3(b). It represents the principal value of the potential (15). The extreme points of each field line have the same azimuth. They mark a cut in the potential surface. By continuation of the potential surface to azimuth angles <0 or $>2\pi$ the potential becomes many valued.

Figure 4(a) shows a 2D field generated by a source and a vortex at the same position. The pattern of field lines, of course, resembles the flow of water running out of a nearly empty bathtub. The corresponding potential surface [Fig. 4(b)] resembles a snail shell in the region around its axis.

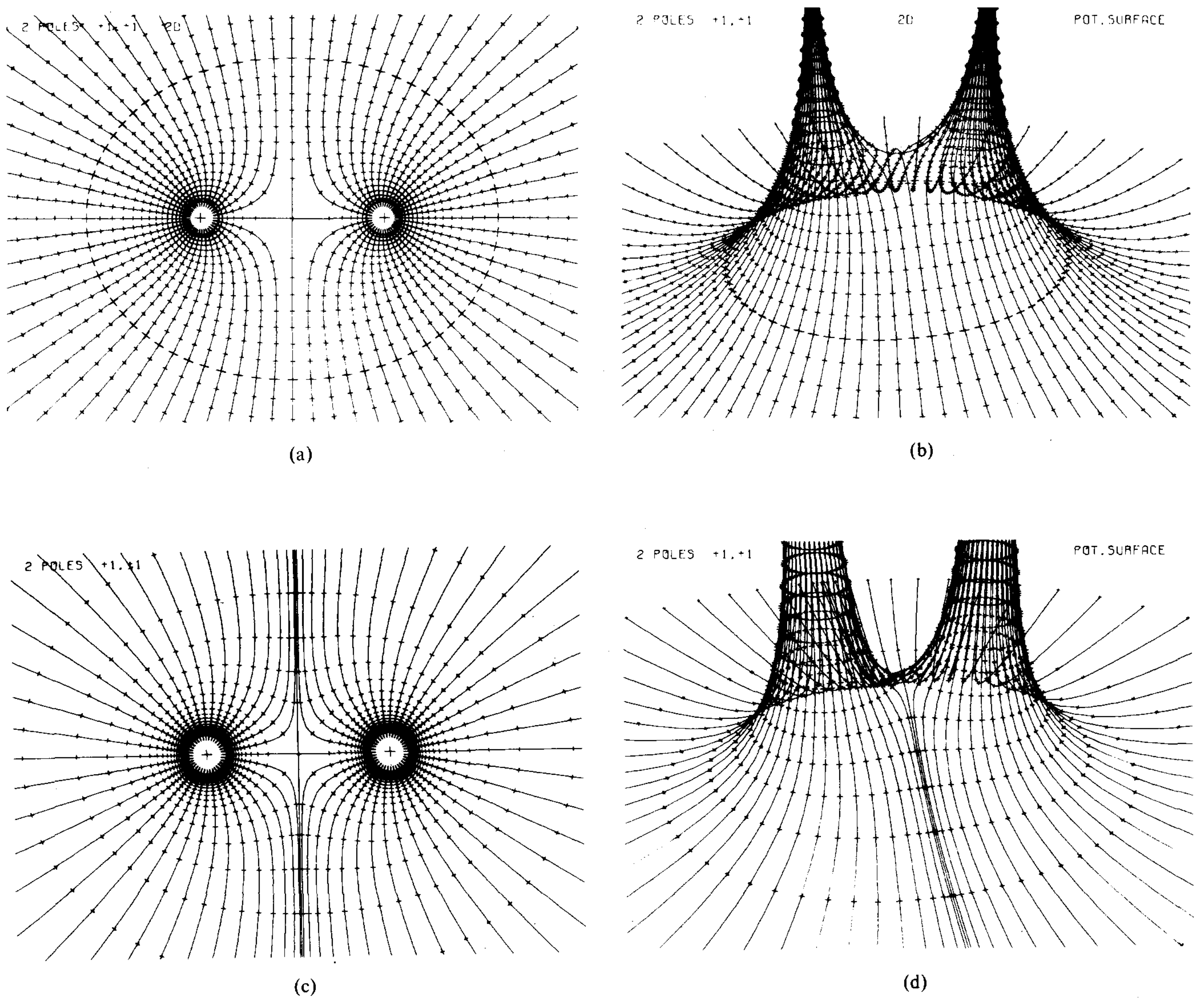


Fig. 5. Field lines in the x,y plane of two identical sources ($q_1 = q_2 = 1$). (a) Two-dimensional (2D) field, $x_1 = -x_2 = 0.5$, $\Delta\phi \approx 1/36$ ($\Delta\phi$ was chosen 7‰ less than $1/36$, in order to place a potential mark on the saddle point). (b) Potential surface of two-dimensional (2D) field, observer's azimuth $\alpha_{obs} = -101°$, $F_\phi = 1/2$. (c) Three-dimensional (3D) field, $x_1 = -x_2 = 0.5$, $\Delta\phi = 1/8\pi$. (d) Potential surface corresponding to (c), $\alpha_{obs} = -101°$, $F_\phi = 1/2$.

B. Two and more poles

The 2D field generated by two identical sources is shown in Fig. 5. Figure 5(a) is a picture of field lines; Fig. 5(b) is a perspective view of the potential surface. If again we identify the field with the velocity field of a liquid, we see that the liquid flows away from both sources but that liquid never crosses a straight line which divides the plane symmetrically. This straight line is itself a field line, which however does not originate from a source. It can nevertheless be drawn by our program using the following "trick." The field lines do not begin with azimuth 0, $\Delta\alpha$, $2\Delta\alpha, \ldots$, but with ϵ, $\Delta\alpha + \epsilon$, $2\Delta\alpha + \epsilon, \ldots$, where ϵ is a very small angle ($\epsilon = 0.001°$ for a 3D and $0.0001°$ for a 2D field). Thus the field line starting from the left-hand source to the right-hand runs slightly above the line connecting both sources. Since in the midpoint of the connecting line the field has a saddle point, near that point the field line changes its direction by practically 90° and now runs very nearly upwards in Fig. 5(a). Similarly, the field line running from the right-hand source towards the left-hand changes its direction and runs downwards. The central point is distinguished by the fact that the field vanishes there. Under these conditions two (or even more) field lines and potential lines are allowed to cross. In this way the singular straight field line is in fact composed of two halves. Since the central field line is straight, the right-hand (and the left-hand) part of the picture can also be interpreted as the field lines of a single source placed near an impermeable wall. The characteristics of the field become very clear by looking at the potential surface in Fig. 5(b). Figures 5(c) and 5(d) illustrate the field of two equal sources in the 3D case.

Comparison of Fig. 5(c) with Fig. 5(a) again shows that field lines and equipotential lines no longer form squares since liquid can now flow "into the third dimension," i.e., perpendicular to the paper plane. This is conspicuous for the field lines near the horizontal axis through the sources: the liquid flowing within a narrow cone from one source towards the symmetry plane can spread in all directions along this plane and the field lines originally bounding the cone for continuity reasons come very near to the plane.

The 3D field of a source and a sink of equal strength (q_1

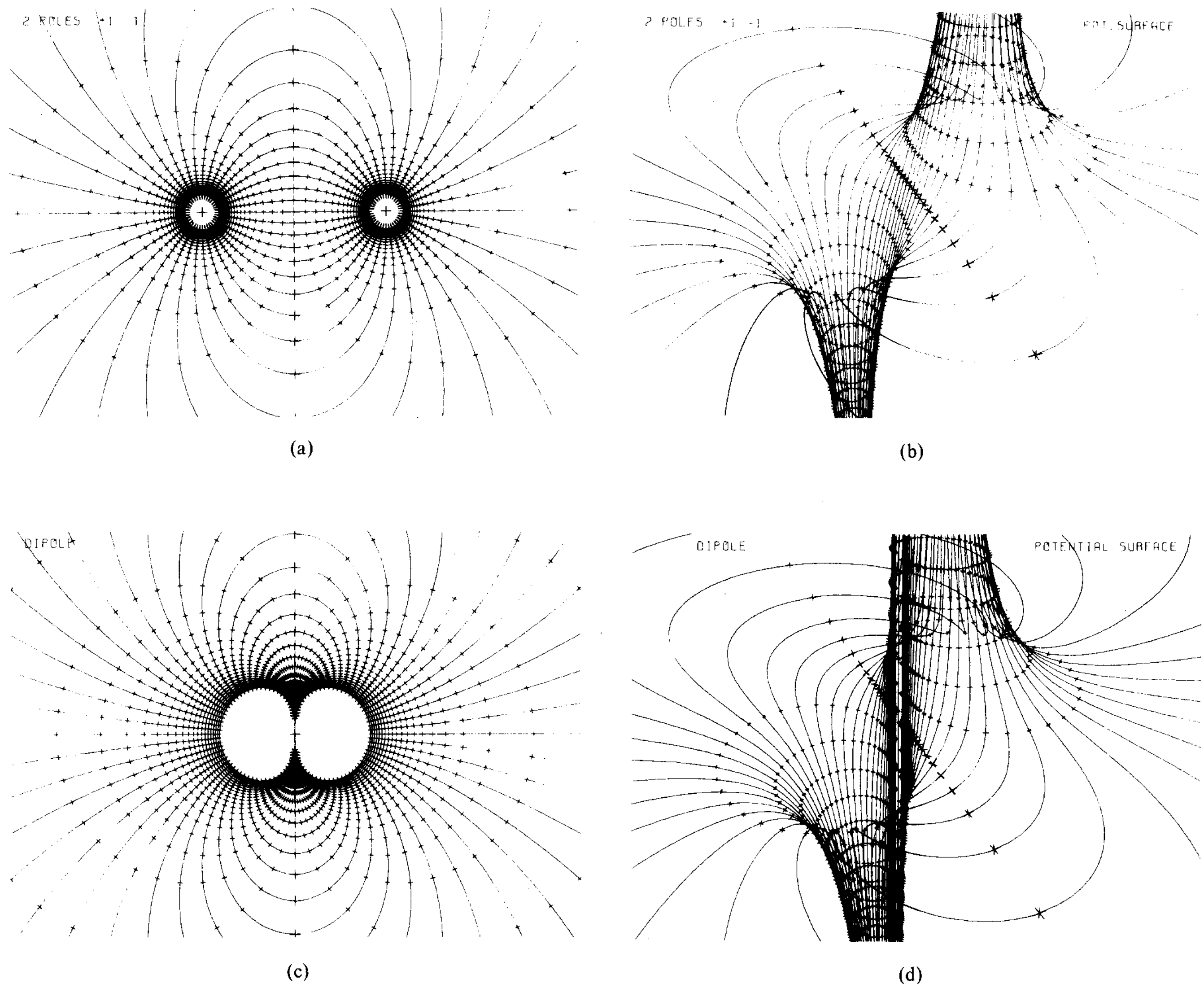


Fig. 6. Three-dimensional (3D) field in the x,y plane of two sources of opposite strength. (a) $q_1 = -q_2 = 1$, $x_1 = -x_2 = 0.5$, $\Delta\phi = 1/8\pi$. Large crosses mark $\phi = 0$. (The crosses drawn for marking the positions of the charges should *not* be interpreted as being the signs of the charges.) (b) Potential surface corresponding to (a), $\alpha_{obs} = -110°$, $F_\phi = 1$. (c) Dipole of strength $d = 1$ and orientation in x direction at the origin, $\Delta\phi = 1/8\pi$. (d) Potential surface of a dipole as in (c), $\alpha_{obs} = -110°$, $F_\phi = 1$.

$= -q_2$) is illustrated in Figs. 6(a) and 6(b). All the liquid flows from the source (on the right) to the sink. The symmetric line between both is now not a field line but an equipotential line ($\phi = 0$). Again the structure of the field can most easily be seen from the potential surface. The symmetry line divides the field into regions of positive (right half-plane) and negative (left half-plane) potential. At infinity the potential approaches $\phi = 0$.

By decreasing the distance a between the two sources but keeping the product $d = aq$ constant in the limit $a \rightarrow 0$, one obtains the field of a dipole of moment d. The corresponding field lines and potential surface are shown in Figs. 6(c) and 6(d). The symmetry line is an equipotential line. The potential undergoes a discontinuity from $-\infty$ to $+\infty$.

An impression of the field in space generated by two sources of opposite charge, i.e., a spatial representation of the field of Fig. 6(a), is given in Figs. 7(a) and 7(b). They are a pair of stereo pictures of the field lines between one of the two sources and the symmetry plane between them. This field pattern would also arise if a single charge were placed in front of a metal surface. The position of the sources and of the eyes of the observer are given in the figure caption. The field lines are seen through the symmetry plane. The potential marks are 3D crosses with one axis in the direction of the field. Using the crosses at the end points of the field lines, it can easily be seen that the field is perpendicular to the symmetry plane. Since the field lines were started symmetrically from the source, the end points mark concentric circles on the plane.

Different fields generated by four sources placed on the corners of a square are shown in Fig. 8 for four equal sources and in Fig. 9 for sources of equal strength but alternating sign. The most prominent difference between the 2D field [Figs. 8(a) and 8(b)] and the section through the 3D field [Figs. 8(c) and 8(d)] is that for continuity reasons there can be no liquid flow towards the center in the 2D case. In the 3D case liquid can flow towards the center and then "escape" into the third dimension. Near the center the flow tubes make a sharp turn out of the paper plane. This can also be seen from the potential surfaces. Only in the 3D case is there a depression of the potential near the center. If the sources are positive electric charges, the motion of a free

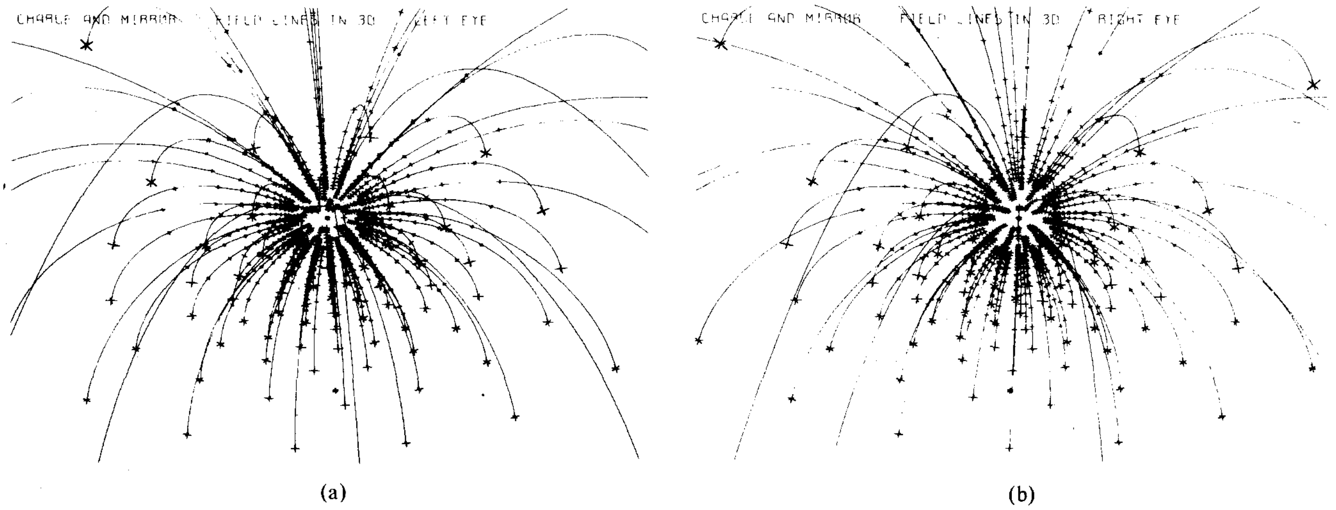


Fig. 7. Field lines in three dimensions between a charge ($q = 1$, $x = y = 0$, $z = 1$) and a metal surface with potential $\phi = 0$ in the x,y plane. There are 92 field lines starting isotropically from the charge. The highest potential on the lines is $\phi = 1/2\pi$. The potential interval between the marks is $\Delta\phi = 1/20\pi$. (a) and (b) are stereoscopic views. The observer positions are: (a) $x = 5.06$, $y = 4.17$, $z = -4.59$ (left eye); (b) $x = 4.43$, $y = 4.83$, $z = -4.59$ (right eye).

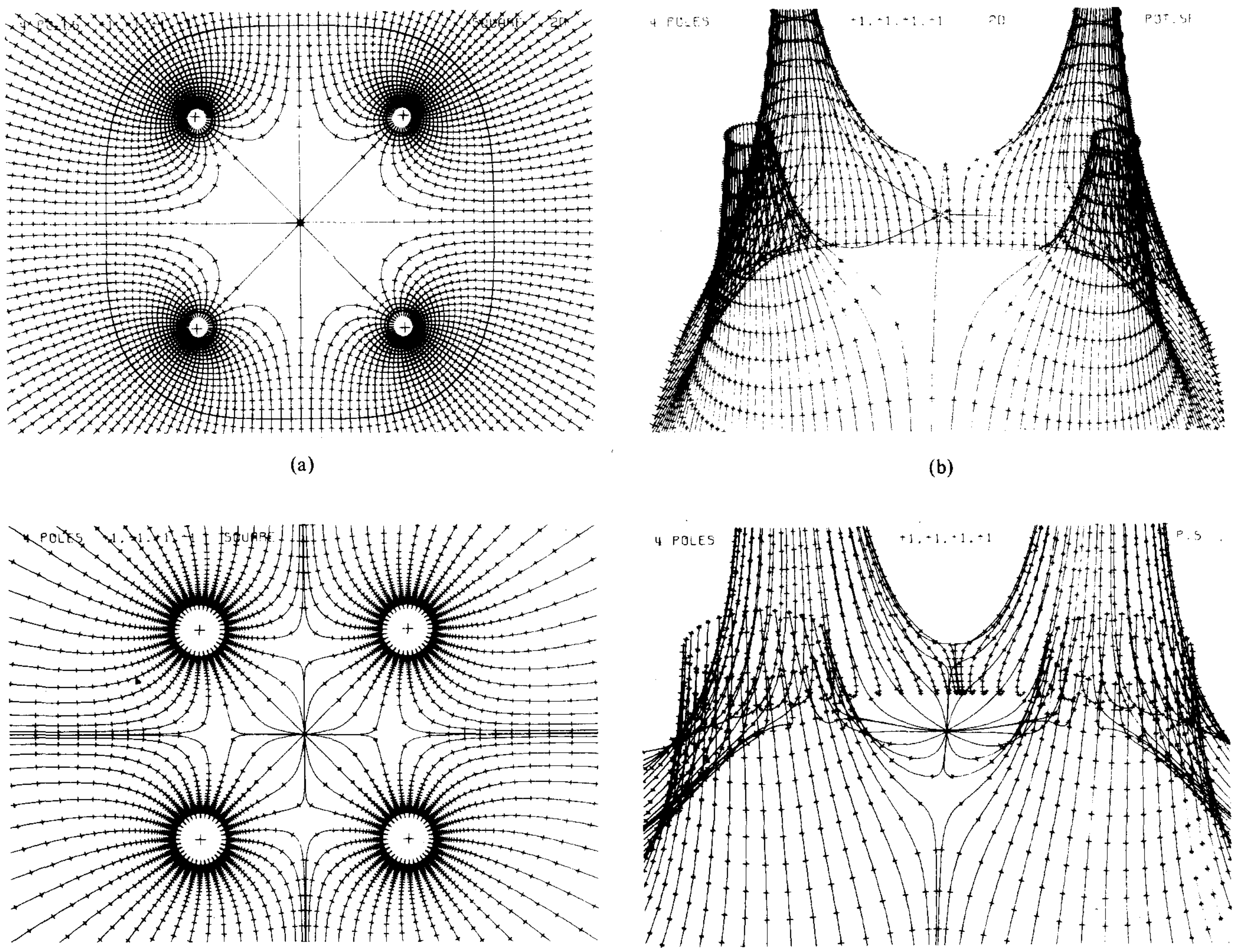


Fig. 8. Field of four equally strong sources ($q = 1$) placed on a square around the origin in the x,y plane ($x_1 = x_4 = -x_2 = -x_3 = 0.5$, $y_1 = y_2 = -y_3 = -y_4 = 0.5$). (a) Two-dimensional (2D) field, $\Delta\phi \approx 1/40$. (b) Corresponding potential surface, $\alpha_{obs} = -87.5°$, $F_\phi = 1/2$. (c) Three-dimensional (3D) field. Field lines in the x,y plane, $\Delta\phi = 0.2933/4\pi$. The innermost equipotential line consists of two closed curves that intersect each other. (d) The corresponding potential surface, $\alpha_{obs} = -87.5°$, $F_\phi = 1/2$.

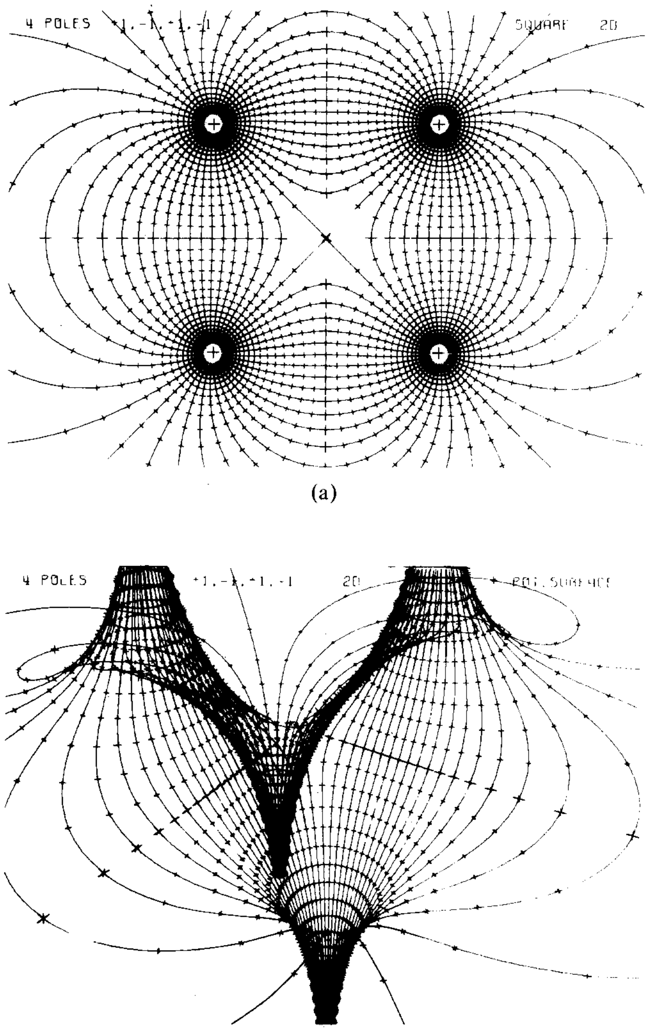


Fig. 9. Field of four sources with equal strength ($|q| = 1$) but alternating sign, placed on a square around the origin in the x,y plane ($x_1 = x_4 = -x_2 = -x_3 = 0.5, y_1 = y_2 = -y_3 = -y_4 = 0.5$). (a) Two-dimensional (2D) field, $\Delta\phi = 1/40$. (b) The corresponding potential surface, $\alpha_{obs} = -58°$, $F_\phi = 1/2$.

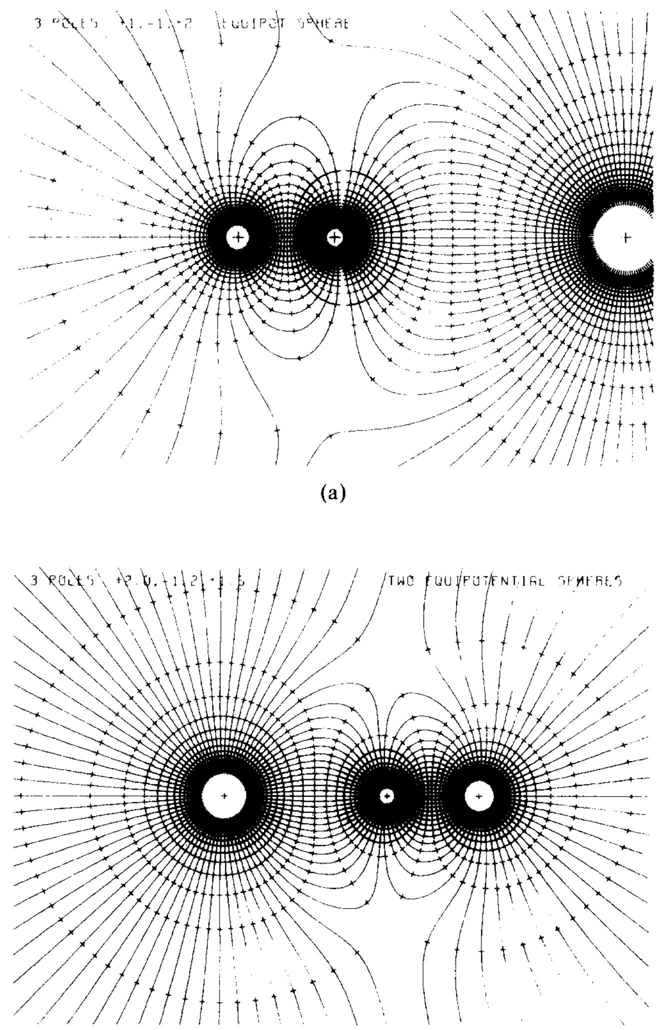


Fig. 10. Field lines in the x,y plane of the three-dimensional (3D) field generated by three charges situated on the x axis. (a) $q_1 = 1, q_2 = -1, q_3 = 2$; $x_1 = 0, x_2 = 0.5, x_3 = 2$. The interval between equipotential surfaces is $\Delta\phi = 1/8\pi$. For $\phi = +1/4\pi$ the equipotential surface is a sphere. It is the surface extending furthest to the left. (b) $q_1 = 2.0, q_2 = -1.2, q_3 = 1.5$; $x_1 = -1.6, x_2 = 0, x_3 = 0.9$, $\Delta\phi = 1/8\pi$. At $\phi = +1/4\pi$ there are two spherical equipotential surfaces (those extending furthest to the left and furthest to the right) which intersect orthogonally.

positive charge can be confined to a limited region around the center, provided it can be kept in the plane of the sources.

C. More complex fields

The 3D field generated by two special configurations of three charges situated on a straight line is shown in Figs. 10(a) and 10(b). These fields are described in the textbook by Durand (Ref. 3, Vol. I, pp. 115 and 117, respectively). Each of the two fields possesses a distinguished equipotential surface.

In the field of Fig. 10(a) there exists a circular equipotential line, i.e., a spherical equipotential surface in space. The field outside that surface could also have been obtained by replacing the two sources at the left by an insulated conducting sphere in the place of the spherical equipotential surface.

In Fig. 10(b) two orthogonally intersecting spheres of different radius form an equipotential surface which surrounds all three charges. The field outside that surface can also be realized by constructing a metallic surface from two truncated spheres and placing on it the total electric charge.

The superposition of a homogeneous field and the field of a negative and a positive point charge of equal strength is given in Fig. 11. Comparison with Fig. 6 shows that the field lines originating from the point sources are now confined to a limited region around the sources. This region is bounded by a singular field line. In the potential plane the extreme points of this region are saddle points.

So far in this section we have discussed fields generated by sources and a homogeneous field. We conclude it with two figures involving vortices.

Figure 12 shows the two-dimensional field of a single source and a single vortex. The field lines which start from the source are distorted by the presence of the vortex. Near the vortex there is a region with closed field lines. In the potential plane picture this region is represented by a helical

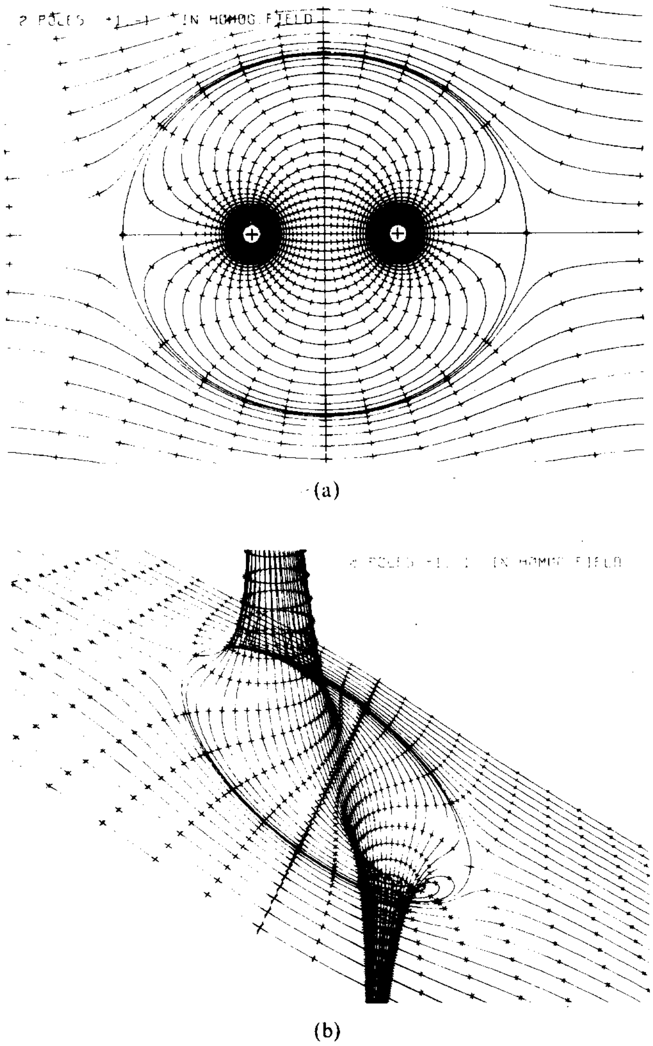


Fig. 11. (a) Three-dimensional (3D) field of two equal charges of opposite sign placed in a homogeneous field, $q_1 = -q_2 = 1$, $x_1 = -x_2 = -0.5$, $y_1 = y_2 = 0$, $E_x = 1/4\pi$, $E_y = E_z = 0$, $\Delta\phi = 0.39688/4\pi$. (b) Corresponding potential surface, $\Delta_{obs} = -75°$, $F_\phi = 1$.

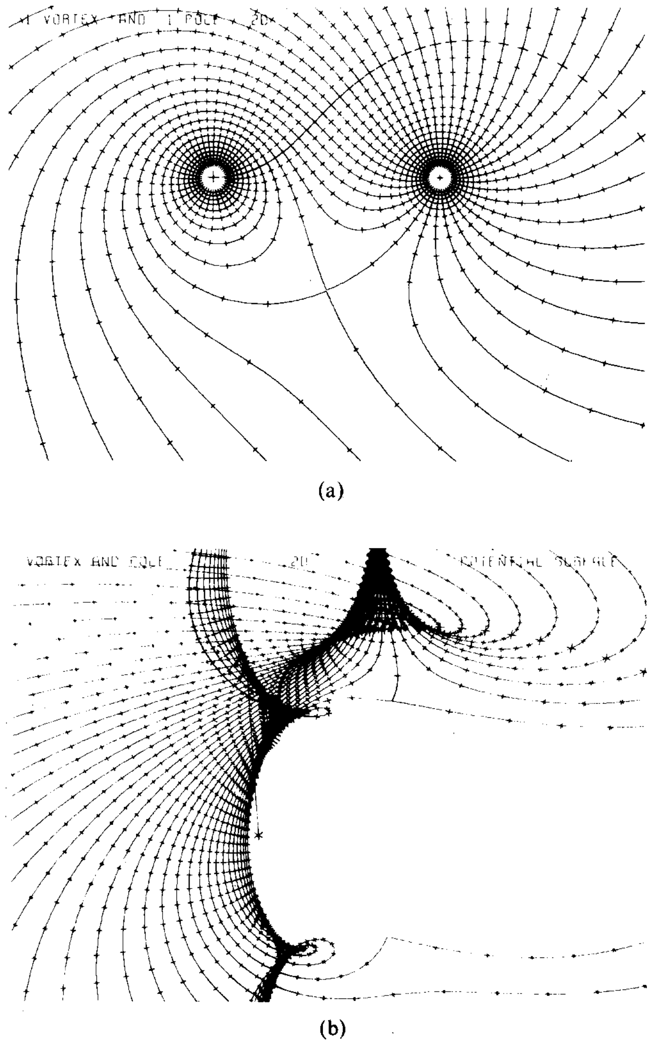


Fig. 12. (a) Two-dimensional (2D) field of a source ($q = 1, x = 1, y = 0$) and a vortex ($\omega = 1, x = 0, y = 0$), $\Delta\phi = 1/36$. (b) Corresponding potential surface, $\alpha_{obs} = -140°$, $F_\phi = 1/4$. In x,y projection the lowest field line at the upper right coincides with the upper field line at the bottom right.

surface, which is embedded into the potential surface dominated by the pole. The total potential has a discontinuity line extending from the vortex to the bottom right of the picture. In Fig. 12(b) this line is drawn once for the lower and once for the upper potential.

The field of a single vortex in a homogeneous field is represented in Fig. 13. Such a field arises if a current-carrying wire is placed in a homogeneous magnetic field or a rotating cylinder in a homogeneous flow field.

D. Fields and particle trajectories

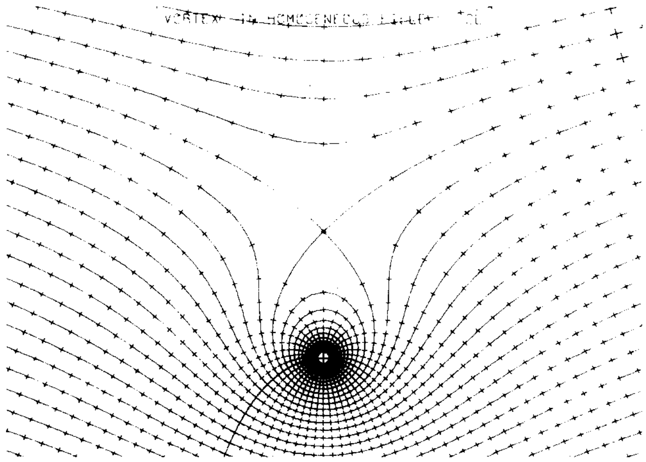
The program allows field lines and the trajectories of particles in the field to be drawn at the same time; likewise, it can display particle trajectories on potential surfaces. Examples are shown in Figs. 14 and 15.

The orbit of a positively charged particle around two fixed negative charges is shown in Fig. 14(a). The initial conditions were chosen in such a way that the particle motion is confined to a plane. The potential surface for that plane is reproduced in Fig. 14(b) together with the particle trajectory on the surface. The picture represents a mechanical analog of the motion of a charged particle in an electric field, namely, the motion of a little ball on a rigid object of the form of the potential surface under the influence of gravity. One observes that the particle is fast in regions of low potential and slow where the potential is high (transformation from kinetic to potential energy).

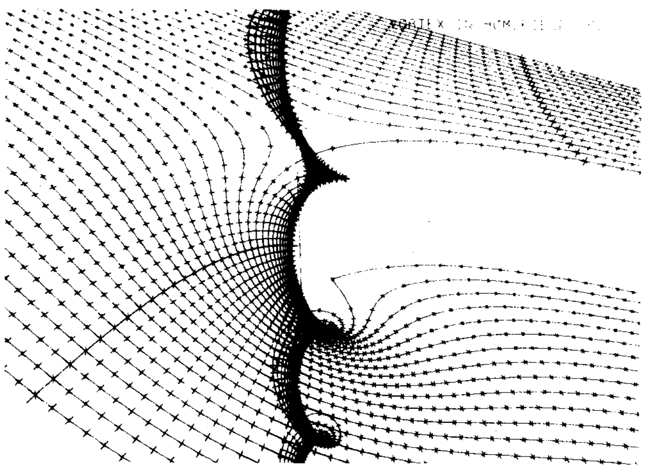
Another example is the motion of an electrically charged particle in the field of a magnetic dipole. In a magnetic field the particle experiences a Lorentz force perpendicular to the field and to its own velocity. Figure 15 shows how a charged particle can be "trapped" between converging magnetic field lines. It can serve as an illustration of the Van Allen radiation belt, which consists of charged particles trapped in the earth's magnetic field.

VI. POSSIBLE EXTENSIONS

The method of interactive computer construction of field lines and equipotential surfaces can be extended in several

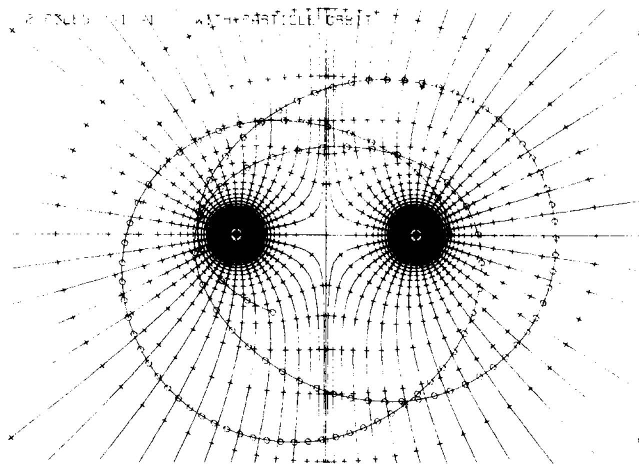


(a)

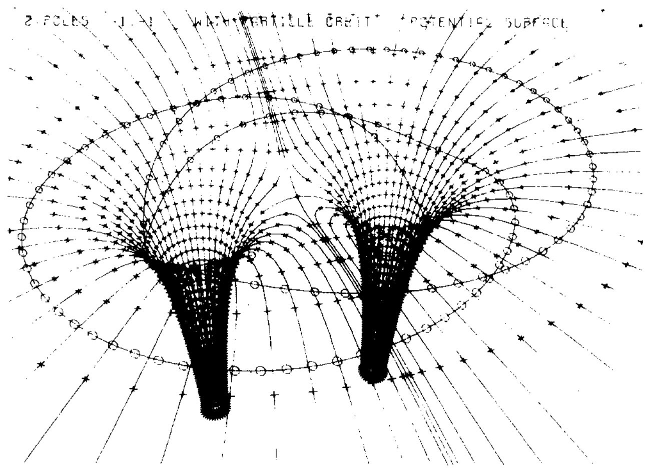


(b)

Fig. 13. Superposition of a two-dimensional (2D) homogeneous field of strength $E_x = 1/2\pi$ in the y direction and a vortex at the origin having $\omega = 1$. The zero level of the potential (large crosses) would coincide with the y axis if the vortex were absent. (a) Field lines in the x,y plane, $\Delta\phi = \frac{1}{36}$. (b) Potential surface, $\alpha_{\rm obs} = -75°$, $F_\phi = \frac{1}{4}$. In x,y projection the lowest field line at the upper right is the same as the highest field line at the bottom right.



(a)



(b)

Fig. 14. Motion of a positively charged particle ($q = 4\pi$) of unit mass in the field of two negatively charged poles ($q_1 = q_2 = -0.125$, $x_1 = -x_2 = 0.5$, $y_1 = y_2 = 0$). The difference between equipotential marks is $\Delta\phi = 1/64\pi$. The particle's initial conditions are $t_0 = 0$, $x_0 = 0.6845$, $y_0 = 0$, $\dot{x}_0 = 0$, $\dot{y}_0 = 1.145$. The particle is shown as a small sphere on its trajectory after time intervals $\Delta t = 0.2$. (a) Field lines and trajectory in the x,y plane. (b) Potential surface and trajectory in x,y ϕ space, $\alpha_{\rm obs} = -105°$, $F_\phi = 2$.

ways. We mention two examples which are of interest for both physicists and mathematicians.

A. Solutions of potential equations

So far in our program only those fields are used which can be simply computed by superposition of a homogeneous field and the fields of sources, vortexes, and dipoles. For the case of an arbitrary charge distribution in space the field has to be found by solving the potential equation for that distribution, usually under given boundary conditions. Programs for solving such problems numerically (by the method of finite differences) exist. Difficulties arise for vortices and dipoles and for field areas extending to infinity. There are problems in distributing field lines, i.e., finding appropriate starting points, at least in the 3D case. However, at least for a restricted class of problems it is possible to extend the existing programs to include the solution of potential equations.

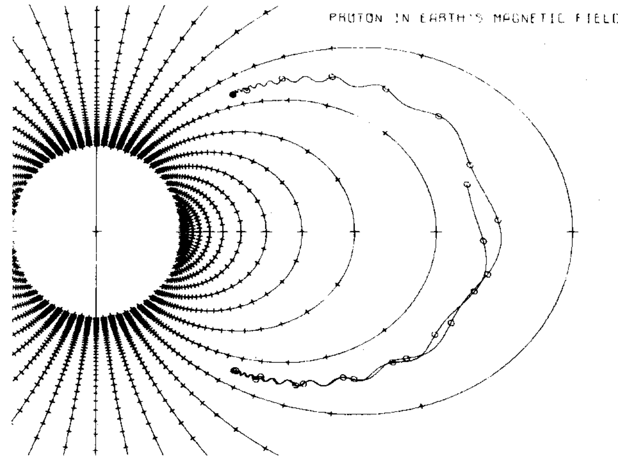


Fig. 15. Trajectory of a charged particle in a magnetic dipole field. The field lines are started on a circle—not on an equipotential line like in Fig. 6(c). The particle starts at the full black circle. Its starting direction is perpendicular to the picture plane. About three-fourths of a period of the oscillation between the northern and southern hemispheres is drawn.

B. Conformal mapping

For 2D fields one can formally identify the Cartesian coordinates x and y with the real and imaginary parts of a complex number $z = x + iy$. An analytic function $w(z) = \phi + i\psi$ has the property that $\phi(z)$ and $\psi(z)$ simultaneously satisfy the Cauchy–Riemann differential equations $\partial\psi/\partial x = -\partial\phi/\partial y$, $\partial\psi/\partial y = \partial\phi/\partial x$. By repeated partial differentiation they yield $\nabla^2\psi = \nabla^2\phi = 0$. Because of these properties (orthogonality of lines of constant ϕ and lines of constant ψ, fulfillment of the Laplace equation) lines of constant ψ can be identified with field lines and lines of constant ϕ with equipotential lines. A uniform field in the x direction, which is described by a rectangular grid of ψ and ϕ lines, is thus represented by the analytic function $w = \text{const}\cdot z$. Other fields can often be obtained elegantly from the uniform field by a conformal transformation, the field line picture from the rectangular grid by conformal mapping (see, e.g., Refs. 3–5).

By incorporating this technique into our program it would be possible to familiarize the student with the connection between electrostatics and the theory of functions.

ACKNOWLEDGMENT

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Deep in the human unconscious is a pervasive need for a logical universe that makes sense. But the real universe is always one step beyond logic.

–Frank Herbert