Notes on the implementation of the inner MHD boundary in the LFM code

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1 Calculation of the cell volume in the LFM code.

What is said in this section is true for the calculation of the grid metric quantities for the code in general, not only for the inner boundary. The calculation described here is valid for any hexahedron mesh. In the legacy LFM fortran code the cell volume calculation is grid dependent and is thus not as general. The parallel version of LFM uses new method. The corresponding code can be found in metric.F. This method is also used in the new C++ calculation of the field-aligned current.

The method for the calculation of the volume of a general hexahedron cell (faces are not necessarily planar) is given by [*Coakley*, 1981]. In this approach the cell volume is just given by the triple product of the vectors connecting the centers of the opposite faces of the cell. Here, the position vector of the center of a face is defined as the average of the position vectors of the four corresponding vertices. A more general (and more accurate) formulation is given by [*Kordulla and Vinokur*, 1983] and later improved by [*Davies and Salmond*, 1985].

References

Coakley, T. J. Numerical method for gas dynamics combining characteristic and conservation concepts, AIAA Paper 81-1257, 1981.

Davies, D. E., Salmond, D. J., Calculation of the volume of a general hexahedron for flow predictions, *AIAA Journal*, Vol. 23, No. 6, pp. 954–956, 1985.

Kordulla, W., M. Vinokur, Efficient Computation of volume in flow predictions, *AIAA Journal*, Vol. 21, No. 6, 1983.

2 Calculation of the current density.

The current calculation is based on the vector identity:

$$\int_V dV\, \nabla\left(\vec{A}\otimes\vec{B}\right) = \int_{\partial V} \vec{B}\left(\vec{A}\cdot d\vec{s}\right), \tag{1}$$

where V is a region in space with boundary ∂V, the integral on the left-hand side is a volume integral of the dyadic product of vectors $\vec{A}$ and $\vec{B}$, and the integral on the right-hand side is

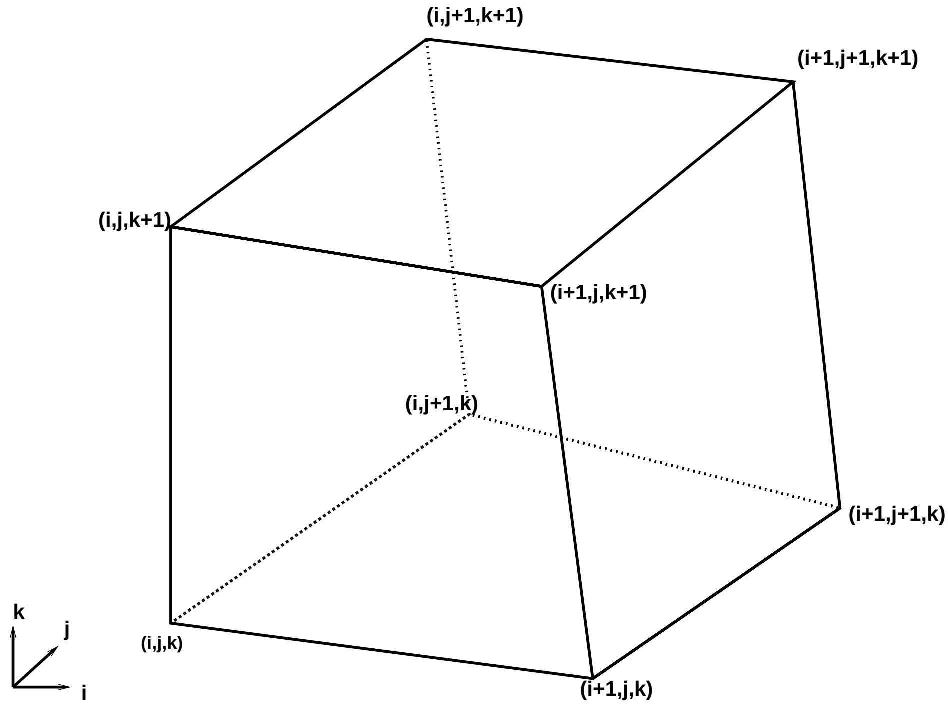

Figure 1: Cell stencil.

the surface integral over the boundary ∂V. Using $\vec{A} = \vec{j}$ and $\vec{B} = \vec{x}$, where $\vec{j}$ and $\vec{x}$ are the current density and position, one obtains

$$\int_V dV\, \nabla \left(\vec{j} \otimes \vec{x} \right) = \int_{\partial V} \vec{x} \left(\vec{j} \cdot d\vec{s} \right). \tag{2}$$

The integral on the left hand side can be simplified:

$$\int_V dV\, \nabla \left(\vec{j} \otimes \vec{x} \right) = \int_V dV\, \left[\vec{x} \nabla \vec{j} + \left(\vec{j} \nabla \right) \vec{x} \right] = \int_V \vec{j}\, dV, \tag{3}$$

where we have used $\nabla \vec{j} = 0$ and $\left(\vec{j} \nabla \right) \vec{x} = \vec{j}$. Therefore, from Eq. (2) it follows

$$\int_V \vec{j}\, dV = \int_{\partial V} \vec{x} \left(\vec{j} \cdot d\vec{s} \right). \tag{4}$$

The approximated Eq. (4) written for a given grid cell yields

$$\left\langle \vec{j} \right\rangle V_{cell} = \sum_\alpha \vec{x}^c_\alpha \int_{S_\alpha} \vec{j} \cdot d\vec{s}, \tag{5}$$

where $\left\langle \vec{j} \right\rangle$ is the average current density in the cell, V_{cell} is the cell volume, the summation on the right-hand side is carried out over the cell faces, $\vec{x}^c_\alpha$ is the center of the corresponding face, i.e. the average of position vectors of the four face vertices, and S_α denotes that the integral is taken over the corresponding cell face. In this notation the area element vector $d\vec{s}$ is defined to point outward of the cell. Further, taking advantage of the Ampere's Law

$$\mu_0 \int_S \vec{j} \cdot d\vec{s} = \oint_L \vec{B} \cdot d\vec{l}, \tag{6}$$

where L is the boundary of surface S, from Eq. (5) one obtains

$$\left\langle \vec{j} \right\rangle = \frac{1}{\mu_0 V_{cell}} \sum_\alpha \vec{x}^c_\alpha \oint_{L_\alpha} \vec{B} \cdot d\vec{l}, \tag{7}$$

where L_α is the edge of quadrilateral α and the integrals over the opposite faces are taken in opposite directions, since the corresponding area element vectors were defined to point outward of the cell.

Fig. 1 shows a general hexahedron cell. We will call the left and the right faces i-face and (i+1)-face. Respectively, the front and the back faces are called j-face and (j+1)-face, and the top and the bottom faces are called k-face and (k+1)-face. In this notation Eq. (7) becomes

$$\left\langle \vec{j} \right\rangle = \frac{1}{\mu_0 V_{cell}} \left[\left(\vec{x}^c_{i+1} I^{int}_{i+1} - \vec{x}^c_i I^{int}_i \right) + \left(\vec{x}^c_{j+1} I^{int}_{j+1} - \vec{x}^c_j I^{int}_j \right) + \left(\vec{x}^c_{k+1} I^{int}_{k+1} - \vec{x}^c_k I^{int}_k \right) \right], \tag{8}$$

where $I^{int}_{i,j,k}$ stand for the line integrals in Eq. (7) calculated for the corresponding faces.

3 Staggered mesh

The implemented calculation of the current density is accomplished on a staggered mesh. Fig. 2 shows a slice of the grid at some k+1/2. The black squares denote the centers of the actual grid cell. So, the calculation described above is actually done on the offset grid drawn in red. The current is therefore specified at the centers of the red cells (red crosses). In the case of the calculation in the two innermost shells the current is specified at the i=2 layer. In the two outermost points (j=1 and j=njp1) with a black circle around the red crosses the current is set to zero. In the rest of the points (red crosses) the current is calculated according to Section 1.

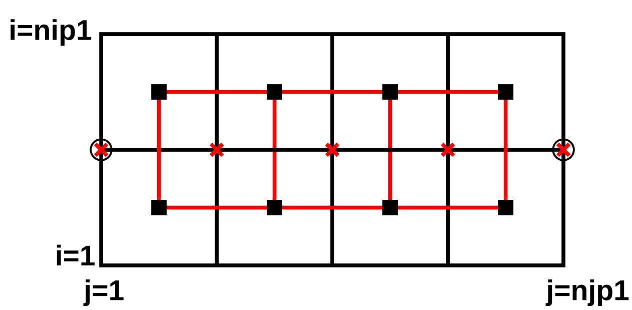

Figure 2: Slice of the grid at some k+1/2.

4 Ionospheric field aligned current

The method for the calculation of the current density described above is used in the LFM code to calculate the current density in the innermost shell of the magnetospheric grid. In order to obtain the field aligned current flowing in and out of the ionosphere the above current should be mapped along the field lines. Let the magnetospheric current density be $\vec{j}_m$ and the mapped ionospheric current density be $\vec{j}_{ion}$. Then

$$j_{\parallel ion} = \frac{\vec{j}_{ion} \cdot \vec{B}_{ion}}{B_{ion}}$$
$$j_{\parallel m} = \frac{\vec{j}_m \cdot \vec{B}_m}{B_m}$$

and using $j_\parallel / B = const$ along a dipole field line one obtains

$$j_{\parallel ion} = \frac{B_{ion}}{B_m} \left(\vec{j}_m \cdot \frac{\vec{B}_m}{B_m} \right), \tag{9}$$

where $\vec{j}_m$ is the current density calculated using Eq. (8). Magnetic field $\vec{B}_m$ is approximated as a dipole field

$$\vec{B}_m = -B_0 R_E^3 \frac{3\left(\hat{n} \cdot \hat{z}\right)\hat{n} - \hat{z}}{R^3}, \tag{10}$$

where $\hat{n}$ is the unit vector in the direction of the point at which the magnetic field is calculated, $\hat{z}$ is the unit vector in the direction of the dipole magnetic moment, B_0 is the magnetic field at the magnetic equator, and R_E is the earth radius. Therefore, the components of the unit vector $\vec{B}_m / B_m$ are given by

$$b_x = -\frac{1}{\sqrt{1 + 3\left(\frac{z}{R}\right)^2}} \frac{3xz}{R^2} \tag{11}$$
$$b_y = -\frac{1}{\sqrt{1 + 3\left(\frac{z}{R}\right)^2}} \frac{3yz}{R^2} \tag{12}$$
$$b_z = \frac{1}{\sqrt{1 + 3\left(\frac{z}{R}\right)^2}} \left[\frac{3\left(x^2 + y^2\right)}{R^2} - 2 \right], \tag{13}$$

where $R^2 = x^2 + y^2 + z^2$. To calculate the ratio B_{ion}/B_m in Eq. (9) one recalls that the equation of the magnetic field line is $R = LR_E \sin\theta$, where θ is the magnetic colatitude, $\cos\theta = z/R$. Therefore,

$$\frac{R_m}{R_{ion}} = \frac{\sin^2 \theta_m}{\sin^2 \theta_{ion}} \tag{14}$$

and

$$\cos^2 \theta_{ion} = 1 - \frac{1 - \cos^2 \theta_m}{R_m / R_{ion}}, \tag{15}$$

where subscript 'ion' denotes the ionospheric footprint of the field line, and 'm'- the point on the field line at the magnetospheric boundary. Finally, using Eq. (10) one obtains

$$\frac{B_{ion}}{B_m} = \left(\frac{R_m}{R_{ion}}\right)^3 \sqrt{\frac{1+3\left[1-\frac{1-\left(\frac{z}{R}\right)^2}{R'_m}\right]}{1+3\left(\frac{z}{R}\right)^2}}, \tag{16}$$

where $R'_m = R_m/R_{ion}$ is the distance to the point 'm' in earth ionosphere radii. Now, using Eqs. (11), (12), (13), and (16) one calculates the ionospheric field aligned current from Eq. (9). Note, that the resulting current density does not depend on the earth's dipole moment.

Finally, the current density calculated using Eq. (8) should be converted to MKS units ($\mathrm{A/m^2}$). In the C++ jpara calculation the coordinates x, y, z are in ionospheric radii and the magnetic field is in the LFM code units (Gs). Therefore, the current given by Eq. (8) before division by μ_0 is in [$\mathrm{Gs/R_{ion}}$] and the conversion factor is

$$\frac{10^{-4}[\mathrm{T/Gs}]}{\mu_0 \cdot rion \cdot 10^{-2}[\mathrm{m/cm}]} = \frac{10^5}{4\pi \cdot rion} \frac{[\mathrm{T/Gs}]}{[\mathrm{H/m}][\mathrm{m/cm}]},$$

where μ_0 is $10^{-7}/4\pi$ [H/m] and $rion = 6.5 \cdot 10^8$ is the ionosphere radius in cm.

5 Notes on the legacy LFM code implementation

In the legacy LFM code, magnetic field B_{ion} in Eq. (8) is approximated to be equal to the field at the pole and constant over the entire polar cap. Using Eq. (10), where $\hat{n} = \hat{z}$ and $n = 1$, one obtains $B_{ion} \approx 2B_0$. In the current version of LFM $B_{ion} = 0.55$ Gs (see function *jsetup* in ionosphere.F). Magnetic field $\vec{B}_m$ given by Eq. (10) is calculated by functions *bxqq0, byqq0, and bzqq0* in dipole.F. Further, in the LFM $B_0 = 0.3$ Gs (see definition of *geoqmu* in INPUT1 file). Later *geoqmu* is redefined (see inital.F in OMP code and init-fortran.F in P++ code) as $B_0 R_E$ [Gs cm^3]. In the LFM magnetic field units are Gs and distance units are cm. Therefore, the current conversion factor is $10^{-2}/\mu_0 = 10^5/4\pi$. Finally, from Eq. (9) follows

$$j_{\parallel ion}[\mathrm{A/m^2}] = \frac{0.55 \cdot 10^5}{4\pi} \frac{\vec{j}_m[\mathrm{Gs/cm}] \cdot \vec{B}_m[\mathrm{Gs}]}{B_m^2[\mathrm{Gs^2}]} = 4370.77 \frac{\vec{j}_m[\mathrm{Gs/cm}] \cdot \vec{B}_m[\mathrm{Gs}]}{B_m^2[\mathrm{Gs^2}]}. \tag{17}$$

In the LFM the constant used is 4376.76 (see function *jsetup* in ionosphere.F).